Quality Factor Concept
in Piezoceramic Transformer Performance Description

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Abstract – A new general approach based on the quality factor concept in piezoceramic transformer (PT) performance description is proposed. The system’s quality factor, material elastic anisotropy, and coupling factor of the input and output sections of an electrically excited and electrically loaded PT fully characterize the resonance and near-resonance PT behavior. A maximum efficiency, transformation ratio, and input and output power were analytically analyzed and simulated as functions of the load and frequency for simplest classical Langevin-type and Rosen-type PT designs.

A new formulation of the electrical PT input impedance allows to separate the power consumed by PT from the power transferred into the load. The system’s PT quality factor takes into account losses in each PT input-output-load functional components. The PT loading process is changing its input electrical impedance on the way that under loading minimal series impedance is increasing and maximal parallel impedance is decreasing coincidently. The quality-factors ratio, between the state of non-loaded and fully-loaded PT, is one of the best measure of PT’s dynamic performances. Practically, the higher the ratio between non-loaded and fully-loaded PT quality factors is, the better PT efficiency. A simple and effective method for the loaded PT quality factor determination is proposed. As was found, a piezoceramic with high piezoelectric anisotropy is required to provide maximum PT efficiency and higher corresponding voltage gain. Limitations on the PT output voltage and power, caused by non-linear piezoceramic material behavior, were established.

Keywords: Rosen-type and Langevin-type piezoelectric transformer, longitudinal and transverse vibration mode, resonance power conversion, high power, nonlinear piezoceramics properties

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I. INTRODUCTION

Small electronic devices which operate at high voltages require a compact transformer to step up the low voltages of available power supplies. The study of PTs has become a very active research area in engineering. In the literature, many PTs have been proposed [1-5] and a few of them, mostly a multilayer Rosen-type PT [6-8], found practical applications, such as electronic ballasts for fluorescent lamps [9], battery chargers, adapters and others. They have simple structure and are constructed from a single small size piece of piezoelectric ceramic [10], and are therefore relatively easy to manufacture.

A Rosen type PT (Fig. 1) typically is operating in the length extensional vibration mode at its first harmonic frequency such that one full standing wave exists on the bar. It consists of a long (2L) thin bar of piezoelectric ceramic which is poled (\( \vec{P} \)) to saturation in the thickness (H) direction along one half (the input, or driving section), and poled along the length direction along the other half (the output, or generating section). It causes the left piezoelement to deform when the voltage is applied whereas the right constructive element will be deformed due to the mechanical resonance of the system. Accompanying the resonance high mechanical strain will be electric fields due to the piezoelectric effect. Thus, the length poled portions of the bar will develop a voltage differential along the length of that section proportional to its length, so that the voltage step-up ratio of the PT will be proportional (open-circuit) to the ratio of the length of the bar to its thickness (or layer thickness in multilayer design). The PT plate is supported at optimal points with minimum strain.

![Fig. 1. A PT base plate with an n-multilayer input section.](image)

Typical PTs convert up to 10 W electrical energy at a high efficiency larger than 95% and have a high voltage step-up ratio of about two to three orders in magnitude depending on PT construction, piezomaterial and load. PTs, like resonators, are classified according to their mode of vibration; there are symmetric and asymmetric PT constructions, voltage and current PTs.
PTs can be described as mechanical continuum structures by *analytical equations* with additional degrees of freedom for the electrical behavior [11]. Analytical models help to understand the mechanisms of vibration generation and transfer. The main problem of these models is that many simplifications are needed to find a closed analytical solution. The other two major approaches to analyze PTs include *equivalent-circuit* (EC) method [12,13] and *finite element method* (FEM) [14,15]. Using FEM formulation for PT, it is difficult to consider load PT effects. On the other hand, even though PT EC model is simple to consider PT load effects, this model has difficulty in finding equivalent-circuit parameters, such as capacitances, inductances, and resistance of a PT. Traditional EC represents the behavior of the PT under the assumption that the mechanical impedances of the two halves of the PT are matched. Mechanical losses in the material are represented by the real component of the wave number which is describable in terms of the mechanical quality factor.

A traditional course during the PT development procedure includes [16] identification of power electric requirements; choice of rough geometry and operating principle using simple analytical continuum models; detailed analysis of the electromechanical behavior with equivalent models; detailed construction supported by finite element models.

In general, the PT performance is strongly load dependent [17,18]: the gain increases with increasing load and reaches a maximum at infinite load (open-circuit), power peaks at a moderate load, and efficiency peaks at a relatively low load. For applications that demand high efficiency, the PT is limited to operate within a small load range that corresponds to low values (not maximal) of power transfer and voltage gain.

The gains at infinite load are not realized for loads less than about $1\, G\Omega$. An inspection of a gain versus load curve showed that at relatively small loads the slope of the gain curve is nearly constant [18].

It was particularly found that for the near maximum efficiency regime of PT operation the gain slope is inversely proportional to the PT plate length and does not depend on the material quality factor $Q_m$.

Piezoelectric ceramic transducers, which are usually driven at a high level of vibrations in a resonant mode with a strong electric field and/or a high mechanical stress [19-21], require optimizing their performance. Material parameters becomes significantly nonlinear with thresholds in their electric field and stress dependences, while typical PT analyses were given only for the low voltage levels of excitation.
Adverse effects of the *electromechanical losses* in PT include undesirable *heat generation*, sub-optimal electrical matching, and reduced operational efficiency. The PT’s efficiency is governed by the loss mechanisms (mechanical, electrical, and piezoelectrical) in the piezoceramic [22]. It is necessary for piezoelectric ceramics for high power use to have [2,8,9,23,24] high coupling coefficients, high mechanical $Q_m$ (including strong fields), high coercive field and high mechanical strength, etc. Consequently, PTs are typically “high Q” devices, made from hard piezoceramics [10]. There is a varying amount of mechanical and dielectric loss in the piezoceramic at all frequencies, their effects are more significant at some frequencies and operational regimes than at others. The dielectric loss typically increases a PT’s resistive component, not voltage gain.

When piezomaterials are driven at higher drive levels the thermal conditions can change considerably under high level of excitation, which in turn depends on PT electrical load. Theoretical models of the steadystate temperature field in a bar-shaped PT operating in longitudinal vibration mode were developed in [25,26] and used to estimate the effects of the ambient, size and heat dissipation conditions on the temperature rise of the PT. The maximum in the internal PT loss distribution corresponds to the PT regions with maximal stress near the PT resonance. As was experimentally shown, the maximum temperature of the PT under excitation is located in its driving part. The heat loss to ambient air is essentially dependant on the PT surface area to volume ratio and the internal thermal loss in PT. The steadystate temperature of a thin long PT plate is proportional along with material dissipative and piezoelectric parameters, to the dielectric permittivity, input electric field strength squared, and the ratio of thickness to length of the plate.

In the present work the following consecutive approaches are used: establishing the PT system’s quality factor relationship with the PT electrical load, establishing the frequency PT characteristics relationship with the PT load, and finally linking them both into PT quality factor relationship with PT’s frequency characteristics under wide range of load variation, including dissipative and electro-mechanical nonlinear PM parameters for “strong field” excitation levels determined on standard samples during piezomaterial characterization and certification.
2. LANGEVIN-TYPE PT

A. Basic Description.

A simple 1-D model of the classical Langevin-type PT [4,27] is considered in the chapter. As shown in Fig. 2, its active piezoelectric part consists of two piezoceramic rods, each with length $L$, width $w$, and thickness $h$. The first section with respective electrodes is poled along thickness and the second section is poled along length (asymmetric voltage step-up combined Rosen-Langevin-type PT). The two loading masses $M$ are connected on the two flat faces of the rods, and they are taken sufficiently large in order to lower the working frequency of the simplest length extensional mode, and consequently to provide a homogeneous mechanical and electrical field distribution avoiding space integration. With this model, the elastic and dielectric parameters, electromechanical coupling factors (CEMC) and quality factor fully characterize the PT resonance voltage-transfer function, energy conversion and accumulation - all the transfer functions of the system can be analytically derived without approximations.

![Fig. 2. A simplified Langevin-type PT model with two coupled relatively large masses $M$ and two non-inertial piezo-sections between them.](image)

Material losses in the PT are taken into consideration by the employment of complex values for the piezoceramic material constants [22]:

$$
\hat{s}_{ij}^{E,D} \equiv s_{ij}^{E,D}(1-i\gamma_{ij}), \quad \hat{d}_{kl} \equiv d_{kl}(1-i\gamma_{kl}), \quad \hat{\varepsilon}_{mn}^{T} \equiv \varepsilon_{mn}^{T}(1-i\delta_{mn}),
$$

(1)

where $i=\sqrt{-1}$, $Q_{ij}^{E,D}$ are the quality factors of the complex elastic compliances $s_{ij}^{E,D}$, $\delta_{mn}$ and $\gamma_{kl}$ are the factors of dielectric (complex permittivity $\varepsilon_{mn}^{T}$) and piezoelectric (complex piezoelectric coefficient $\hat{d}_{kl}$) losses,
respectively. Real and imaginary parts of these material parameters are provided by the manufacturer of the piezoelectric materials and can be found in the literature [10,16,22]. Complex values for the piezoelectric constant $\hat{d}_{kl}$ can be identified in experiments [28] but they are not independent of the dielectric and elastic loss components, and have to satisfy restrictions, to make sure no energy would be generated in the material [22,28].

According to the particular geometry, electrical and mechanical boundary conditions, the PT can be described by 1-D scalar constitutive equations [22]:

For input PT section 1:

\[
\varepsilon_{(1)} = \left( u_0 - u_{M(i)} \right) / L , \quad \varepsilon_{(1)} = s_{11}^{E} T + \hat{d}_{31} E_{x(1)} , \quad D_{3(1)} = \varepsilon_{33}^{T} E_{x(1)} + \hat{d}_{31} T , \quad I = -D'_{3(1),t} \cdot Lw = -i\omega D_{3(1)} \cdot Lw , \quad I = V \cdot Y_{in} , \quad V = -E_{3(1)} \cdot h \quad (2)
\]

\[
M u_{M(i),u}^{*} = T \cdot h w , \quad -\omega^2 M u_{M(i)} = T : h w
\]

For output PT section 2:

\[
\varepsilon_{(2)} = \hat{s}_{33}^{E} T + \hat{d}_{33} E_{x(2)} , \quad D_{3(2)} = \varepsilon_{33}^{T} E_{x(2)} + \hat{d}_{33} T , \quad J = D'_{3(2),t} \cdot h w = i\omega D_{3(2)} \cdot h w , \quad U = R \cdot J , \quad U = -E_{3(2)} \cdot L \quad (3)
\]

\[
M u_{M(2),u}^{*} = -T \cdot h w , \quad \omega^2 M u_{M(2)} = T : h w
\]

where $u_{M(i)}$ is the mass displacement, $u_0$ is the PT input-output boundary displacement, $\varepsilon_{(i)}$ and $T$ are the mechanical strain and stress in the piezoceramic body, $E_{3(\theta)}$ and $D_{3(\theta)}$ are the electric field strength and displacement, $\omega = 2\pi f$ is the frequency, $R$ is the load resistance, $Y_{in}$ is the PT input impedance, $I$ and $J$, $V$ and $U$ are the PT input and output currents and voltages, respectively, $i = 1,2$ is the PT section number, material indexes used correspond to the traditional notations [22], variables’ time dependence is taken as $e^{i\omega t}$, a single layer PT is considered in this chapter for simplicity.

Due to the momentum conservation $u_M \equiv u_{M(i)} = -u_{M(2)}$, then

\[
E_{3(2)} = -\frac{U}{L} = -\frac{\hat{d}_{33}}{\varepsilon_{33}} \cdot T \cdot \frac{i\hat{f}}{1 + i\hat{f}} , \quad \varepsilon_{(2)} = \hat{s}_{33}^{T} T \left( 1 - \kappa_{33}^2 \frac{i\hat{f}}{1 + i\hat{f}} \right) , \quad u_M = -LsT \left( \frac{\omega}{\omega_r} \right)^2
\]

\[
T = -\frac{\hat{d}_{31}}{2s} \left( \frac{V}{h} \right) \frac{\left( \omega / \omega_r \right)^2}{1 - \left( \omega / \omega_r \right)^2} \cdot \hat{s} / s = -\frac{\hat{d}_{31}}{2s} \left( \frac{V}{h} \right) \frac{\left( \omega / \omega_r \right)^2}{1 - \left( \omega / \omega_r \right)^2} \left( 1 - i/Q \right) , \quad (4)
\]
where \( \omega_c = h w / s M L \), \( \omega_r = 2\pi f_r \) is the PT resonance frequency, \( \hat{\omega} = \omega \hat{C}_2 R \) is the complex relative loading parameter, \( \hat{C}_2 = \varepsilon^{T}_{33} w h / L \) is the complex output PT capacitance, \( \hat{k}_{33} = d_{33}^{2}/\varepsilon_{33}^{T} \hat{\varepsilon}^{E} \) is the complex longitudinal CEMC squared,

\[
\hat{s} = \frac{1}{2} \left[ \hat{s}_{11}^{E} + \hat{s}_{33}^{E} \left( 1 - \hat{k}_{33}^{2} \frac{i \hat{t}}{1 + i \hat{t}} \right) \right] \equiv s \left( 1 - i \frac{1}{Q} \right) \tag{5}
\]

is the system’s vibrational (dynamic) complex compliance with its real component \( s \) and the system’s vibrational quality factor \( \hat{Q} \) according to:

\[
s = 0.5 \left( s_1 + s_2 \right), \quad s_1 = s_{11}^{E}, \quad s_2 = s_{33}^{E} \left( 1 - k_{33}^{2} \frac{t^2}{1 + t^2} \right), \tag{6}
\]

where \( s \) is the total dynamic PT compliance, \( s_1 \) is the compliance of the input PT section, \( s_2 \) is the dynamic compliance of the output PT section, which changes from \( \min s_2 = s_{33}^{E} \left( 1 - k_{33}^{2} \right) = s_{33}^{D} \) for the open-circuited (o.c.) output PT section up to \( \max s_2 = s_{33}^{E} \) for the short-circuit (s.c.) condition,

\[
\frac{1}{\hat{Q}} = \frac{1}{Q_{(m)}} + 0.5 B_2 k_{33}^{2} \cdot \frac{t}{1 + t^2}, \tag{7}
\]

\[
\frac{1}{Q_{(m)}} = \frac{B_1}{2Q_{41}^{E}} + \frac{B_2}{2} \left\{ \frac{1}{Q_{33}^{E}} - k_{33}^{2} \cdot \left( 2\gamma_{33}^{E} - \delta_{33(\text{out})}^{E} \right) \right\} \frac{t^2}{1 + t^2} = \frac{B_1}{2Q_{41}^{E}} + \frac{B_2}{2} \left\{ \frac{1}{Q_{33}^{E}} - \left( \frac{1}{Q_{33}^{E}} - \frac{1 - k_{33}^{2}}{Q_{33}^{D}} \right) \right\} \frac{t^2}{1 + t^2}, \tag{8}
\]

where \( k_{33}^{2} = d_{33}^{2}/\varepsilon_{33}^{T} s_{33}^{E} \) is the real longitudinal CEMC squared, \( t = \omega C_2 R \) is the real relative loading parameter, \( C_2 = \varepsilon_{33}^{T} w h / L \) is the real output PT capacitance, \( B_1 = s_{11}^{E}/s \) and \( B_2 = s_{33}^{E}/s \) are the coefficients of PT elastic anisotropy, \( Q_{(m)} \) is the extended net piezomaterial quality factor determined from the basic material compliances relationships \( s_{33}^{E,D} = s_{33}^{E,D} \left( 1 - i Q_{33}^{E,D} \right) \), \( s_{33}^{D} = s_{33}^{E} \left( 1 - k_{33}^{2} \right) \) with [28,29]

\[
\frac{1}{Q_{33}^{D}} = \frac{1}{Q_{33}^{E}} - k_{33}^{2} \left( 2\gamma_{33}^{E} - \delta_{33(\text{out})}^{E} \right) \frac{1}{Q_{33}^{E}} \tag{9}
\]

The first term \( Q_{(m)}^{-1} \) in (7) characterizes losses inside the PT, and the second term includes losses in the PT
load – generally they are both load-dependable. The expression (7) for the system’s resonance vibrational quality factor $\tilde{Q}$ includes the elastic losses in both PT sections and dielectric and piezoelectric losses is the output section, connected to “vibrational” PT properties. It does not include the dielectric loss $\delta_{33(m)}$ in the input section as it is not a “vibrational” PT characteristic and does not influence the PT voltage gain.

As known [28], the piezoelectric loss caused by the $\gamma_{31}$ ($\hat{d}_{31}$) factor, for the transverse vibrational mode (input section) equals to zero at the system’s resonance due to a 90°-phase shift between the input exciting electric field strength $V/h$ and mechanical stress $T(4)$ at the resonance.

According to [28-30], the resonance $Q_r$ and antiresonance $Q_a$ quality factors of an elementary [31] rod resonator ($k_{33}$-type longitudinal vibration) at its fundamental harmonic are close to $Q_a = Q_{33}^0$ and $Q_r = Q_{33}^E$. At the same time, the quality factors $Q_{r,a}$ have similar behavior in strong field, so that the $Q_(m)$ parameter in (7) will be further considered close to material mechanical quality factor $Q_m$ with its well known properties [10,19,29] determined on standard piezoceramic samples [31]. It is not a restricted consideration – if $Q_{ij}^{E,D}$ are known, they can be taken into account (7,8).

The coefficients $B_1$ and $B_2$ are related to a PT plate “node” position and depend on the stress distribution along the PT plate length – if the point of minimal displacement (“node”) is located in the input PT section, then a share of $Q_{11}^E$ in the total quality factor must be greater, and vice versa. Note that the resonance PT characteristics, such as maximum mechanical displacement and stress

$$\left|u_m\right|_r = 0.5L \left|d_{31}\right| V \frac{h}{h} \cdot \tilde{Q}, \quad \left|T\right|_r = \left|\frac{d_{31}}{2s}\right| \frac{V}{2s} \cdot \tilde{Q}$$

(10)

are determined by the integral system’s elastic compliance $s$ and PT vibrational quality factor $\tilde{Q}$.

The $\tilde{Q}$-factor is dropping down following electrical load changing with minimum near 10 units at loading parameter $t \rightarrow 1$. It follows that the quality-factors ratio, between the state of non-loaded and fully-loaded PT, is one of the best measure of PT’s dynamic (loading) performance.
**B. Input PT Admittance, Voltage Gain, and Energetic Characteristics**

We will further consider the very-important dynamic (under loading) parameters of a multilayer \((n = H/h)\) PT with input \(C_1 = \varepsilon_{33}^T n^2 L w/H\) capacitance and output PT \(C_2 = \varepsilon_{33}^T H w/L\) capacitance, then

\[
C_1/C_2 = (L/h)^2 = (n L/H)^2.
\]

There are two kinds of frequency dependences under consideration:

PT frequency characteristics for a given condition of PT and load, and PT resonance frequency characteristics, particularly when the PT load changes.

The initial resonant factor in (4) can be represented as

\[
\frac{(\omega/\omega_r)^2}{1 - (\omega/\omega_r)^2 (1 - i/\tilde{Q})} \equiv -i \frac{\tilde{Q}}{1 + i y} \quad \text{in the vicinity of the resonance } \omega_r = 2 \pi f_r \quad \text{through the generalized frequency displacement } y = 2 \tilde{Q} \chi, \]

where \(\chi = f/f_r, -1\) is the resonance frequency displacement. Then, the mechanical stress (4), its absolute value and the resonance value can be expressed as

\[
T \equiv 0.5 i \frac{d_{31}}{s} \frac{V}{h} \cdot \frac{\tilde{Q}}{1 + i y}, \quad |T|(f) = \left| \frac{d_{31}}{2s} \frac{V}{h} \cdot \frac{\tilde{Q}}{1 + i y^2} \right|, \quad |T| = \left| \frac{d_{31}}{2s} \frac{V}{h} \cdot \tilde{Q} \right|. \tag{11}
\]

The input PT power \(P_{in} = 0.5V^2 \Re Y_{in}\) is determined through the input PT admittance:

\[
Y_{in}(f) = \frac{1}{V} i \omega L w \left[ \varepsilon_{33}^T \frac{V}{h} + d_{31} T \right] \equiv i \omega L w \left[ \varepsilon_{33}^T - i \frac{d_{31}^2}{2s} \frac{\tilde{Q}}{1 + i y^2} \right], \tag{12}
\]

\[
\Re Y_{in(r)} = \omega_r c_1 \left( \delta_{33(n)} + 0.5 B_1 \cdot k_{31}^2 \tilde{Q} \right), \tag{13}
\]

then

\[
P_{in}(f) \equiv 0.5 \omega c_1 V^2 \left[ \delta_{33(n)} + 0.5 B_1 k_{31}^2 \tilde{Q} \cdot \frac{1}{1 + y^2} \right], \tag{14}
\]

\[
P_{in(r)} \equiv 0.5 \omega_r c_1 V^2 \left( \delta_{33(n)} + 0.5 B_1 k_{31}^2 \tilde{Q} \right), \tag{15}
\]

where \(k_{31}^2 = d_{31}^2 / \varepsilon_{33}^T s_{11}^E\) is the real transverse CEMC squared. PT loading is changing PT’s input electrical impedance on the way that (under loading) maximal resonance admittance is decreasing, and minimal antiresonance admittance is increasing coincidently (12), following the PT vibrational quality factor
\( \tilde{Q} \) dropping down at \( t \to 1 \). As it is seen from (13-15), the input PT power \( P_{in} = P_d + P_v \) consists of two additive terms of the dielectric loss of the PT input section and vibrational PT losses. The last one is determined by the \( \tilde{Q} \)-factor and includes the elastic losses in both sections, dielectric and piezoelectric losses (7-9) in the output PT section.

The output PT power is determined by \( P_r = 0.5 |U|^2 / R \). Taking into account that the input PT capacitance does not directly affect the PT voltage and power transfer, we can express the PT output characteristics using (4,7) as

\[
U = L \frac{d_{33} T}{\varepsilon_{33}^*} \cdot \frac{it}{1+it} \equiv -0.5 i L \frac{V}{h} \cdot \frac{d_{33} d_{31} \tilde{Q}}{\varepsilon_{33}^*} \cdot \frac{it}{1+it} \cdot \frac{1}{1+iy},
\]

then the voltage gain and output PT power can be expressed as

\[
\left| \frac{U}{V} \right| (f) = 0.5 \frac{L}{h} \frac{d_{33} |d_{31}|}{\varepsilon_{33}^*} \tilde{Q} \cdot \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+y^2}},
\]

\[
\left| \frac{U}{V} \right|_{\text{v(r)}} = 0.5 \frac{L}{h} \sqrt{B_1 B_2} k_3 |k_3| |\tilde{Q}| \cdot \frac{t}{\sqrt{1+t^2}},
\]

\[
P_r (f) = \frac{1}{8} \omega C_1 V^2 \cdot B_1 B_2 k_3^2 k_{31}^2 |\tilde{Q}| \cdot \frac{t}{1+t^2} \cdot \frac{1}{1+y^2},
\]

\[
P_{r(r)} = \frac{1}{8} \omega_{(r)} C_1 V^2 \cdot B_1 B_2 k_3^2 k_{31}^2 |\tilde{Q}| \cdot \frac{t}{1+t^2}.
\]

There are two direct influential factors in the voltage gain \( |V/U|_{(r)} \) dependence: the system’s vibrational \( \tilde{Q} \)-factor and the load-factor for a capacitive PT output and active load character. Note that the output voltage \( |U| (f) \) frequency characteristic does not have the antiresonance peak at all, to the contrary of the PT input admittance characteristic with dielectric and vibrational additive terms (12).

Then, the PT efficiency can be expressed as:

\[
\eta (f) = \frac{P_r}{P_{in}} (f) = 0.5 B_2 k_{33}^2 \cdot \frac{t}{1+t^2} \cdot |\tilde{Q}| \cdot \frac{1}{1+2(1+y^2) \cdot \delta \cdot |\varepsilon_{33(m)}| / B_1 k_{31}^2 |\tilde{Q}|},
\]
\[
\eta_{(r)} = \frac{P_R}{P_{in}} = \left(1 - \frac{\tilde{Q}}{Q_{(m)}}\right) \frac{1}{1 + 2\delta_{33(m)} B_1 k_{31}^2 Q}\,.
\]  

(22)

The internal thermal PT power losses are \( P_h = P_{in} - P_R = P_{in} (1 - \eta) \), then

\[
P_h(f) = 0.5 \omega C_1 V^2 \left(\delta_{33(m)} + \frac{1}{2} B_1 k_{31}^2 \frac{\tilde{Q}^2}{Q_{(m)}} \frac{1}{1 + \gamma^2}\right).
\]  

(23)

Taking into account the \( \tilde{Q} \)-factor expressions (7,8), the total thermal power \( P_{h(r)} = P_{h1(r)} + P_{h2(r)} \), where the internal thermal losses in the input \( P_{h1(r)} \) and output \( P_{h2(r)} \) PT sections at resonance are

\[
P_{h1(r)} = 0.5 \omega_c C_1 V^2 \left(\delta_{33(m)} + \frac{1}{4} B_1^2 k_{31}^2 \frac{\tilde{Q}^2}{Q_{(m)}}\right),
\]  

(24)

\[
P_{h2(r)} = \frac{1}{8} \omega_c C_1 V^2 B_1 B_2 k_{31}^2 \tilde{Q}^2 \left[\frac{1}{Q_{33}} - \left(\frac{1}{Q_{33}^D} - \frac{1}{Q_{33}^E} - \frac{k_{33}^2}{Q_{33}^E}\right) \frac{t^2}{1 + t^2}\right].
\]  

(25)

As typically \( Q_{33}^D > Q_{33}^E \equiv Q_{11}^E \) according to the experimental data [28,29,32], it explains the fact that the input PT section is more heated than the output section [25,26]. Obvious physical limit properties follow from the above expressions: when the material CEMC \( k_{31} = 0 \) of the PT input section, there are no losses at all in PT output section 2, otherwise, when the material CEMC \( k_{33} = 0 \) of the PT output section, the losses are elastic only there.

C. “Near” and Maximum Efficiency Loading Regime \((0.1 < t < 10)\).

The o.c. regime of PT excitation with infinite load is not practical. It is used, particularly, at the initial stage of PT operation to ignite a fluorescent lamp, etc. Then, the maximum efficiency regime of PT loading, or near it, is further required for better PT performance. The “near” and maximum efficiency loading regime is defined here as a \( t \)-range with predominately output losses (more than 90\%) in the total PT system, that corresponds (7) to the condition \( t + t^{-1} \leq 0.05 k_{33}^2 Q_{(m)} \), then \( \min t^{-1} = \max t \geq 10 \) for the typical material quality factor \( Q_{(m)} \geq 500 \) and CEMC \( k_{33} \geq 0.65 \). For the typical range \( t \in (0.1...1...10) \)
considered, where $t = \omega_{(r)} C R = \pi R e^{T_{33}}^E \cdot w H L^2$ is the relative load, $v^E = (\rho s_{11}^E)^{-1/2}$ is the material sound velocity, the system’s vibrational quality factor $\hat{Q} \approx 2(t + t^{-1})/B_k k_{33}^2$ is predominantly load-dependent. Then from (17-25) we can express at resonance

$$\left|\frac{U}{V}\right|_{(r)} \equiv (L/h) \cdot \sqrt{A} \cdot t \sqrt{1 + t^{-2}}$$

$$P_{in(r)} = 0.5 \omega_{(r)} C_1 V^2 \left[ \delta_{33(n)} + A \left( t + t^{-1} \right) \right],$$

$$P_{R(r)} = 0.5 \omega_{(r)} C_1 V^2 A \left( t + t^{-1} \right),$$

$$\eta_{(r)} \equiv \left[ 1 - \frac{2}{B_k k_{33}^2 Q_{(m)}} \left( t + t^{-1} \right) \right] \sqrt{1 + \frac{\delta_{33(n)}}{A \left( t + t^{-1} \right)}},$$

$$P_{h(r)} = P_{in(r)} - P_{R(r)} \equiv 0.5 \omega_{(r)} C_1 V^2 \left[ \delta_{33(n)} + 2 \frac{A}{B_k k_{33}^2 Q_{(m)}} \left( t + t^{-1} \right)^2 \right],$$

$$P_{h1(r)} = 0.5 \omega_{(r)} C_1 V^2 \left[ \delta_{33(n)} + \frac{A}{B k_{33}^2 Q_{11}} \left( t + t^{-1} \right)^2 \right],$$

$$P_{h2(r)} = 0.5 \omega_{(r)} C_1 V^2 \left( t + t^{-1} \right)^2 \left\{ \frac{1}{Q_{33}^E} - \left( \frac{1}{Q_{33}^E} - \frac{1}{k_{33}^2 Q_{m}} \right) \left( 1 + t^{-2} \right)^{-1} \right\},$$

where $B = B_2/B_1 = s_{33}^E/s_{11}^E$ and $A = k_{33}^2/B_1 s_{33}^E$ are the material parameter of elastic and piezoelectric anisotropy which are supposed to correspond to the maximal, or equal, level of poling for both PT sections.

The piezoelectric anisotropy parameter [33] $A = d_{31}^2/d_{33}^2 \equiv \sigma_0^2 \approx 0.16$, where $\sigma_0$ is the Poisson coefficient of non-polarized piezoceramics. To provide PT maximum efficiency and power, maximum $\sigma_0$ and $k_{33}^2 Q_{m}$ parameters are required, so that respective piezoceramic compositions and high-density technologies (hot-pressed ceramics, film technology, etc.) are more desirable for better PT performance.

There are two specific regions in the voltage-gain characteristic presented in Fig. 3:

the first one corresponds to $0.1 < t < 1$ with

$$\left|\frac{U}{V}\right|_{(r)} \equiv \sqrt{A} \cdot L/h,$$
which is dependable only on the material piezoelectric anisotropy factor $A$ and length-to-thickness PT ratio: the second region corresponds to $1 < t < 10$ with

$$\left| \frac{U}{V} \right|_{(e)} \equiv t \cdot \sqrt{A} \cdot L / h \equiv \pi \sqrt{A} \varepsilon_{33}^{E} v^{E} \cdot n R \cdot w / L ,$$

when the voltage gain does not depend on PT thickness [17,18]. Note a relatively weak “strong field” effects influence inside both of these $t$-regions, considered further. The minimum vibrational quality factor value $\tilde{Q}$ in the range of “near maximum efficiency” regime corresponds to

$$\frac{1}{\min Q} \equiv 0.5 k_{33}^{2} \frac{B}{1 + B} \left(1 - k_{33}^{2} \frac{B}{1 + B} \right)^{-1/2} \quad \text{at} \quad t_{(\min)}^{2} = \left(1 - k_{33}^{2} \frac{B}{1 + B} \right)^{-1} \equiv \left(1 - 0.8 k_{33}^{2} \right)^{-1},$$

where $t_{(\min)}$ – parameter is determined by partly “clamped” PT output capacitance and is close to 1.

Fig. 3. Calculated (18) PT resonance voltage gain vs. relative load parameter $t = \omega \omega / C_{2}R$ of the Langevin-type PT with $L / h = 10$ for different levels of excitation. Material parameters used: $k_{33} = 0.65$, $k_{31} = 0.32$, $B = s_{33}^{E} / s_{11}^{E} = 1.30$, $Q_{m0} = 500$, $E_{Q1} = 5 V/mm$.

Fig. 4 a,b shows the load dependence of the calculated input, output and thermal resonance PT power, and PT efficiency. The dielectric loss factor creates a saddle-like efficiency dependence due to its maximum relative influence at $t = 1$, and then it sharply decreases for both $t << 1$ and $t >> 1$. The PT output power
dependence reveals three extremes: two of them correspond to the maximal PT output power, the third extreme point corresponds to the maximum efficiency with minimal PT internal losses $P_{h(r)}$ (23-25), that takes place at $\min \tilde{Q}$ (35), and a local minimum per unit output power. Relative load values for the maximal PT output power are beyond of the maximum-efficiency region. According to (20) for a “weak” level of excitation (Fig. 4b, $V/h = 1V/mm$), the $P_r$ maximum corresponds to $\max \left \{ \tilde{Q}^2 \cdot \frac{t}{1+t^2} \right \}$ with $\tilde{Q}$-expression according to (7) and takes place when the PT output loss equals the losses inside PT (thermal losses). Ten, it corresponds to the condition $t + t^{-1} = 0.5B_z k_{33} Q_{(m)}$ for lower $t_\downarrow$ and upper $t_\uparrow$ relative PT load values according to

$$t_\uparrow = t_\downarrow^{-1} = 0.5B_z k_{33} Q_{(m)} \quad \text{with} \quad \max P_{R(r)} = \frac{1}{16} \omega_{(r)} C_1 V^2 B_z \cdot k_{31}^2 Q_{(m)} .$$

(36)

The maximum output power corresponds to a ~50% level of PT efficiency. As “strong field” of excitation greatly, up to an order, decreases the material quality factor $Q_{(m)}$, then the $t_\uparrow, t_\downarrow$ and $\max P_{R(r)}$ values are excitation level sensitive (chapter 4B, Fig. 4 for $V/h = 100 V/mm$).

---

**Fig. 4 a,b.** Calculated (15-23) input, output, thermal resonance PT power (a), and PT efficiency (b) vs. relative load parameter $t = \omega_{(r)} C_z R$ of the Langevin-type PT with $L/h = 10$ for different levels of excitation. Material parameters used: $k_{33} = 0.65$, $k_{31} = 0.32$, $B = s_{33}^E/s_{11}^E = 1.30$, $Q_{m0} = 500$, $E_{Q1} = 5 V/mm$, $\delta_{33(n),0} = 0.01$, $E_{\delta_1} = 25 V/mm$.
A Langevin-type PT considered is a system with concentrated parameters, not requiring space integration as in the case of a ring PT in its description. Note, that basic Rosen-type PT characteristics differ only in part of some close coefficient, such as $8/\pi^2 = 0.81$ (sinusoidal factor) instead of 1 [34].

As it is obviously seen from (5,6), the PT electrical loading changes the complex PT compliance. Its imaginary part is determined by the system’s quality factor $\tilde{Q}$, and its real part influences the PT resonance operating frequency, both in the range from s.c. to o.c. PT output section. Consequently, the relative load $t$-parameter can serve as a linking factor in the practical dependence $\tilde{Q}(f)$, if two kinds of dependences $\tilde{Q} \leftrightarrow t \rightarrow f$ are predominantly established.

3. FREQUENCY PT CHARACTERISTICS

A. PT Resonance Operating Frequency

To determine the resonance frequency of an asymmetric Rosen-type PT system considered further and its variation under the electrical load, the following model and a simple original procedure are used.

A static deformation of a solid PT plate by an internal force $T$ with equal in magnitude and opposite top-plate displacements $u_M$ allows to determine the resonance frequency of such an inhomogeneous plate.

![Diagram](image)

Fig. 5. A model for the resonance frequency determination of an asymmetric PT plate with a unit cross-section. A particular case of relative compliances $s_2 < s_1$ and internal compressed stress $T < 0$ is shown.

The plate consists of two equal-length parts with different elastic compliances $s_1$ and $s_2$. The following typical for 1-D models suppositions were made: the material density is equal for both plate parts and is
constant during the deformation (transverse effects are neglected), the plate is free and is deformed by an internal force so that the center of mass of the plate does not move due to momentum conservation. There is a point inside the plate with zero displacement which corresponds to a node in a lossless vibration case, while the center of plate mass does not coincide with the “node” (Fig. 5). The difference in the \( s_1 \) and \( s_2 \) compliances is caused by the difference in polarization directions of the two PT sections, so that \( s_1 = s_{11}^E \) for the input PT section and two limit values for the output PT section are supposed to be with \( s_2 = s_{33}^{E} \) for s.c. and \( s_2 = s_{33}^{D} = s_{33}^{E} \left(1 - k_{33}^2 \right) \) for o.c. regimes of electrical loading. For PZT piezoceramics \( s_{33}^{D} < s_{11}^E < s_{33}^{E} \) takes place, or \( \left(1 - k_{33}^2 \right)^{-1} > s_{33}^{E} / s_{11}^E > 1 \) with typical ratios \( s_{33}^E / s_{11}^E \equiv B \equiv 1.3 \) and \( s_{33}^D / s_{11}^E \equiv 0.7 \).

When a compressed internal force \( T (T < 0) \) is applied to the plate, its deformations are described by the following equations of elasticity:

\[
\begin{align*}
    u_0 - u_{M(1)} &= s_1 TL, \\
    u_{M(2)} - u_0 &= s_2 TL
\end{align*}
\]

for PT input section 1 and output section 2, respectively, then \( u_M = u_{M(1)} = -u_{M(2)} \)

\[
\begin{align*}
    u_0 &= 0.5(s_1 - s_2)TL, \\
    u_M &= -0.5(s_1 + s_2)TL.
\end{align*}
\]

The point of zero-displacement is located inside one of the two plate sections, depending on the \( s_1 \) and \( s_2 \) elastic constants. The zero-displacement condition for a plate point is equivalent to fixing the plate at the point without any influence on the plate performance under the deformation. Hence, the resonance frequency of the whole plate equals the resonance frequency of a homogenous bar, fixed on one side, with a length \( l \) from the fixing point to its end (for the half-plate where the “node” is located).

Then, for \( s_2 < s_1 \) (particularly, o.c. PT output section) with the frequency-specifying half-dimension \( l \) on the left side (Fig. 5, section 1) according to \( u_M / (L + x_m) = u_0 / x_m \) follows

\[
\begin{align*}
    x_m &= -0.5L \left(1 - s_2 / s_1 \right) \quad \text{and} \quad l = L + x_m = 0.5L \left(s_2 + s_1 \right) / s_1, \quad \text{then} \\
    f_r |_{x_2 < s_1} &= \frac{1}{4(L + x_m) \sqrt{\rho s_1}} = f_0 \cdot \frac{2}{1 + s_2 / s_1}.
\end{align*}
\]
and finally the resonance frequency of the whole PT plate at the o.c. regime of excitation

\[ f_{r,\infty} = f_0 \cdot \frac{2}{1 + S_{33}^D/S_{11}^E} \]  \hspace{1em} (41)

For \( s_2 > s_1 \) (particularly, s.c. PT output section) with the “node”-point on the right side (Fig. 5, section 2) with

\[ x_m = 0.5L\left(1 - s_1/s_2\right) \quad \text{and} \quad l = L - x_m = 0.5L\left(s_1 + s_2\right)/s_2 \]  \hspace{1em} (42)

\[ f_{r}\bigg|_{s_2 > s_1} = \frac{1}{4(L-x_m)\sqrt{\rho \ s_2}} = f_0 \cdot \frac{2\sqrt{s_2/s_1}}{1 + s_2/s_1} \]  \hspace{1em} (43)

and finally the resonance frequency of the whole PT plate at the s.c. regime of excitation

\[ f_{r,0} = f_0 \cdot \frac{2\sqrt{s_{33}^E/s_{11}^E}}{1 + s_{33}^E/s_{11}^E} \]  \hspace{1em} (44)

where \( f_0 = \left(4L\sqrt{\rho s_{11}^E}\right)^{-1} \) is the basic resonance frequency with \( s_i = s_{11}^E \) at any loading.

Then, the limit resonance frequency interval \( G_{r-r} = f_{r,\infty}/f_{r,0} - 1 \) between the PT output o.c. \( (f_{r,\infty}) \) and s.c. \( (f_{r,0}) \) resonance frequencies can be found for a Rosen-type PT

\[ \frac{1}{1 + G_{r-r}} = \sqrt{B} \left(1 - k_{33}^2 \frac{B}{1 + B}\right) \]  \hspace{1em} (45)

Note that, along with \( s_{33}^D = s_{33}^E \left(1 - k_{33}^2\right) \) relationship, the material parameter of elastic anisotropy \( B = s_{33}^E/s_{11}^E \) is proportional to \( 1 + \left(\sim k_{33}^2\right) \), so that for the unpolarized state of piezoceramics with \( k_{33} \rightarrow 0 \) all considered compliances \( s_{ij}^{E,D} \rightarrow s_0 \) to the compliance of isotropic material.

Fig. 6 shows the resonance interval \( G_{r-r} \) dependence on the parameter of elastic anisotropy \( B \). As it is seen, the basic PT resonance frequencies interval is almost twice lower then that for an elementary rod resonator with a \( k_{33} \)-type vibrational mode (\( \delta_r \) is its resonance frequencies interval), and does not depend on the material CEMC \( k_{31} \) of the input PT section.
Fig. 6. Dependence of the PT resonance frequency interval $G_{r-r} = f_{r=0} / f_{r=1}$ (o.c. – s.c.) on the material elastic anisotropy parameter $B = s_{33}^E / s_{11}^E$ for a set of CEMC $k_{33}$ values. Data for a Rozen-type PT plate (solid lines, (45)) and Langevin-type PT plate (dash line, (48)) are presented. The typical parameters $k_{33} = 0.65$ and $B = 1.3$ (●) are taken for further calculations.

The case with an arbitrary, current $s_2(t)$ for a PT under load can be calculated using the basic model expressions (39-43) with the PT input compliance $s_1 \rightarrow s_{11}^E$ and the dynamic PT output compliance

$$s_2 \rightarrow s_{33}^E \left[ 1 - k_{33}^2 \frac{t^2}{1 + t^2} \right],$$

then the total compliance $s(t) = 0.5 \left[ s_1 + s_2(t) \right]$ with its relative components

$$\alpha \equiv \frac{s_2}{s_1} = B \left[ 1 - k_{33}^2 \frac{t^2}{1 + t^2} \right].$$

Then, the resonance frequency variation under the electric load in the units of the basic frequency $f_0 = 1 / 4L \sqrt{\rho s_{11}^E}$ (coincides with the “bar” resonance frequency) can be expressed as (40,43):

$$f_r = f_0 \frac{2}{1 + \alpha} \quad \text{for } \alpha < 1, \quad \text{and} \quad f_r = f_0 \frac{2 \sqrt{\alpha}}{1 + \alpha} \quad \text{for } \alpha > 1.$$
Fig. 7. Dependence (47) of the PT resonance operating frequency on the PT elastic anisotropy parameter \( \alpha(t) \equiv s_2(t)/s_1 \) of the two PT sections. Solid line - Rosen-type PT plate, dot line - Langevin-type PT plate, dash line points out the regions of strong field excitation influence for Rosen-type PT.

The sections’ compliances are equal \( s_2 = s_1 \), or \( \alpha = 1 \), at \( t = \left\{ k_{33}^2 B/(B-1) - 1 \right\}^{1/2} \approx 1...2 \), that corresponds to the near maximum efficiency region of loading. As it follows from Fig. 7, the PT resonance frequency characteristic \( f_r(t) \) changes with much different slopes from the specific \( f = f_0 \) frequency point in the case of a Rosen-type PT (for a system with “distributed parameters” \( f_r \sim 1/l \sqrt{\rho s} \)) to the contrary to a Langevin-type PT (for a system with “concentrated parameters” \( f_r \sim 1/\sqrt{M s} \sim 2/(1+\alpha) \)) with an even characteristic. In common, the same even dependence character as the last one takes place for symmetric PT designs; the results depend on a relative electrode coverage.

Frequency dependences of the resonance output \( P_{R(r)} \) and input \( P_{in(r)} \) power, and current input PT admittance \( |Y_m|(f) \) for different values of the load \( t \) – parameter are presented in Fig. 8 for a Rosen-type PT,
which finally reflect both sides of the described \( \dot{Q} \leftrightarrow t \rightarrow f \) approach. The presented dependences are in a good agreement with experimental data [2].

One can see that the resonance frequency, which corresponds to the minimum input PT admittance, does not correspond to the frequency of maximum (resonance) transformation ratio, input and output power, PT efficiency. In Fig. 8, the intersection of the relative input power \( P_{i(n)} \) and admittance \( |Y_m(f)| \) corresponds to the frequency of maximum PT conductance \( \text{Re} Y_m(f) \) for a given load \( t \)-parameter.

\[ K = \frac{1}{\pi f_a C V^2} \]

**Fig. 8.** Calculated (14-20) frequency dependences of the normalized PT resonance output \( P_{r(n)} \) and input \( P_{i(n)} \) power, current input admittance \( |Y_m(f)| \) for different values of the load \( t \)-parameter. The Rosen-type PT with \( L/h = 10 \) for \( V = 50 \text{V/mm} \) and material parameters \( k_{33} = 0.65, k_{31} = 0.32, B = 1.3, Q_m = 500, \delta_{33(i),0} = 0.01, E_{Q1} = 5 \text{V/mm}, E_{\delta1} = 25 \text{V/mm} \) are used

**B. PT Characteristic Frequencies**

There is a number of PT characteristic frequencies whose relationships characterize PT properties and piezoceramics conditions. For any given load, the output voltage can be controlled by shifting the frequency
above or below resonance - this is the method used in inverters and converters operating in frequency-shift control mode [35].

The PT performance can be estimated by effective electromechanical coupling factors at both the input and output sections (primary material CEMCs $k_{33}$ and $k_{31}$ are involved) in a “weak” field of excitation. The corresponding resonance frequency intervals are used to evaluate the PT performance, as they can be expressed through the resonance and anti-resonance frequencies under different load conditions which could be determined by a modal analysis.

![Graph showing resonance and antiresonance frequencies](image)

**Fig. 9.** Qualitative load dependence of the characteristic resonance ($f_r, \overline{f_r}$) and antiresonance ($\overline{f_a}$) PT frequencies in the range of s.c. – o.c. electrical loads.

The resonance intervals considered are defined as a resonance-to-resonance frequencies ratio $G_{r-r} = f_{r,\infty} / f_{r,0} - 1$, and as antiresonance-to-resonance frequencies ratios $\{G_0, G_{eff}, G_\infty\} = \overline{f_a} / \overline{f_r} - 1$ for corresponding loads $t = 0, 1, \infty$ through the frequencies of maximum and minimum PT input admittance $|Y_{in}|(f)$. As it follows from the Langevin-type PT model:
\[
\frac{1}{(1 + G_{r-r})} = 1 - k_{33}^2 \frac{B}{1 + B}, \quad \frac{1}{(1 + G_0)} = 1 - k_{31}^2 \frac{1}{1 + B}. \tag{48}
\]

\[
\frac{1}{(1 + G_{\infty})} = 1 - k_{33}^2 \frac{1}{1 + B (1 - k_{33}^2)}, \quad G_{\text{eff}} = \frac{1}{\tilde{Q}_{\max} \eta} \equiv \frac{4}{B_3 (t \equiv 1) k_{33}^2}. \tag{49}
\]

As the resonance intervals are relatively small, we can express \( (1 + G)^{-2} \equiv 1 - 2G \), then

\[ G_{\infty}^{-1} = G_0^{-1} - 2 A^{-1}, \quad A \equiv G_0 / G_{r-r}, \quad G_{r-r} \equiv 0.5 (1 - G_0 / G_{\infty}), \tag{50} \]

while the PT limit intervals

\[ G_0 \equiv 0.5 k_{31}^2 / (1 + B) = 0.5 \delta_{\text{r(bar)}} \quad \text{and} \quad G_{r-r} \equiv 0.5 k_{33}^2 B / (1 + B) = 0.5 \delta_{\text{r(rod)}}. \tag{51} \]

are equal to a half of elementary bar and rod PTs [31] own resonance intervals (while \( G_{\infty} \) is their combination), where the parameter of piezoelectric anisotropy \( A \equiv k_{31}^2 / B k_{33}^2 = \alpha_{31}^2 / d_{33}^2 \equiv \sigma_0^2 = 0.16 \).

Then for extremely low \( \min \tilde{Q} = 10 \), it follows the equality of the maximum efficiency resonance interval \( G_{\text{eff}} \) to \( G_{r-r} \):

\[ G_{\text{eff}} = l / \min \tilde{Q} \equiv t_{\text{min}} \cdot 0.5 k_{33}^2 B / (1 + B) = G_{r-r} \cdot t_{\text{min}} = G_{r-r}. \tag{52} \]

Qualitative load dependences of the resonance and antiresonance PT frequencies are presented in Fig. 9, where \( \bar{f}_r \) and \( \bar{f}_a \) are the PT resonance and antiresonance frequencies determined as frequencies of maximum and minimum of the input PT admittance \( |Y_{in}|(f) \), \( f_r \) is the resonance frequency determined particularly as a frequency of maximum PT voltage gain \( |U/V| \) (input conductance \( \text{Re} Y_{in} \) and PT efficiency \( \eta \), as well).

In common, the two types of the resonance frequencies \( \bar{f}_r \) and \( f_r \) coincides for at least sufficiently large \( \tilde{Q} > 100 \). Note, there is no any antiresonance frequency for the \( |U/V|(f) \), \( \text{Re} Y_{in}(f) \), \( \eta(f) \) current characteristics due to an absence of an additive dielectric component as in the expression (12) for the input PT admittance. Fig. 10 reflects the following property regarding PTs vibrational quality factor in non-loaded and loaded conditions: the resonance frequency interval for the input admittance peak intensities less than 8 dB due to loading corresponds to the reciprocal vibrational quality factor value \( \tilde{Q}^{-1} \), that allows to easily control the basic PT parameter [30].
Fig. 10. Dependence of the relative frequency interval \( G \) and reciprocal system’s vibrational quality factor \( Q^{-1} \) on the input resonance intensity, as a consequence of loading, for a Langevin-type PT under input voltage \( V/h = 1 \text{ V/mm} \). Material parameters used: \( k_{33} = 0.65 \), \( k_{31} = 0.32 \), \( B = s_{33}^E / s_{11}^E = 1.30 \), \( Q_{m0} = 500 \), \( E_{q1} = 5 \text{ V/mm} \), \( \delta_{33(\omega),0} = 0.01 \), \( E_{b1} = 25 \text{ V/mm} \).

C. PT “node” dynamic displacement

Strictly speaking, in the considered real asymmetric PTs under load there is no a node point at all – there is a point of minimal displacement with the same, as the node would have, coordinates. The influence of this point is tremendously important for PT performance, as it works like a piston transferring energy from the input PT section into output one. For the dynamic Langevin-type model, the PT center \( u_0 \) and PT top \( u_M \) (masses) displacements \( u_M = u_{M(1)} = -u_{M(2)} \) can be found from (2-3):

\[
\frac{u_0}{\mu} = \frac{B \left( \frac{\omega}{\omega_r} \right)^2 - 1}{1 - \left( \frac{\omega}{\omega_r} \right)^2 \left( 1 - i/\bar{Q} \right)} - 2, \quad \frac{u_M}{\mu} = \frac{1}{1 - \left( \frac{\omega}{\omega_r} \right)^2 \left( 1 - i/\bar{Q} \right)},
\]

(53)

where \( \mu = 0.5d_{33} L \cdot V/h \) is the unit displacement. Then, at the PT resonance:
\[
\frac{u_0}{\mu} \bigg|_r = i \tilde{Q} \left(\frac{2}{1+\frac{s_2}{s_1}} - 1\right) - 2 \quad \text{and} \quad \frac{u_M}{\mu} \bigg|_r = -i \tilde{Q} .
\]

When statically pressed, the point of zero displacement according to (39,42) is \( x_m/L = 0.5 \left(1 - \frac{s_1}{s_2}\right) \) for \( s_2 > s_1 \), and \( x_m/L = -0.5 \left(1 - \frac{s_2}{s_1}\right) \) for \( s_2 < s_1 \). An estimation gives about \( \max|x_m/L| = 15\% \), which corresponds typically to \( \sim 2...3 \) mm for \( L \) near 20 mm. The “node” resonance displacement

\[ |u_m| = \min |u(x)| = 2\mu \] at the \( x_m \) PT point, and it is in-phase with applied voltage. For this reason, a tough fixing of a Rozen-type (asymmetric) PT is not possible (in the contrary to elementary resonators and symmetric-type PTs). The dependence of the PT displacement distribution of the Langevin-type PT model for three characteristic values of the load \( t \) - parameter is presented in Fig. 11.

![Fig.11](chart.png)

**Fig.11.** Dependence of the relative PT displacement distribution of the Langevin-type PT model for three characteristic values of the load \( t \) - parameter. Material parameters used: \( B = \frac{s_{33}^{E}}{s_{11}^{E}} = 1.30 \), \( k_{33} = 0.65 \), \( Q = 500 \) (“weak” regime of excitation).
4. PHENOMENOLOGICAL ENERGETIC PT PERFORMANCE REPRESENTATION

A. Total PT Quality Factor

A PT gives the power consumed, dissipated and transferred, when a certain sinusoidal field level is applied to a device. A common energy balance of a loaded PT is schematically shown in Fig. 12.

According to the quality factor definition as a ratio $Q = \omega W_{kin}/P$ of the kinetic energy of the system to its losses for a period of vibration [28] at any frequency including resonance, we can derive and then define

$$P_{in} = P_h + P_R = P_{d(1)} + P_{vR} \cdot \quad P_{h} = P_{h(1)} + P_{h(2)} = P_{d(1)} + P_{v} \cdot \quad P_{vR} = P_v + P_R \cdot \quad P_v = P_{v(1)} + P_{v(2)}, \quad (55)$$

$$Q_{sys} = \frac{\omega W_{kin}}{P_{in}} \text{ is the total system's quality factor,} \quad \hat{Q} = \frac{\omega W_{kin}}{P_{vR}} \text{ is the vibrational system's quality factor,}$$
$Q_{(m)} = \frac{\omega W_{kin}}{P_v}$ is the material PT quality factor, $Q^\ast_{(m)} = \frac{\omega W_{kin}}{P_h}$ is the extended material PT quality factor.

Then, the following relationships between the quality factors can be established:

$$\tilde{Q} = \frac{Q_{(m)}}{1 + P_R/P_v}, \quad Q_{sys} = \frac{Q_{(m)}}{1 + P_d/(1 + P_R/P_v)} \quad \text{and} \quad Q^\ast_{(m)} = \frac{Q_{(m)}}{1 + P_d/(1 + P_R/P_v)}.$$  \hfill (56)

According to the PT efficiency definition $\eta = \frac{P_R}{P_{in}} = \frac{P_R}{P_h + P_R} = \frac{P_R/P_v}{1 + P_d/(1 + P_R/P_v)}$, then

$$\eta = 1 - \frac{P_h}{P_{in}} = 1 - \frac{Q_{sys}}{Q^\ast_{(m)}} = \left(1 - \frac{\tilde{Q}}{Q_{(m)}} \right) \left(1 + \frac{P_d}{P_v} \cdot \frac{\tilde{Q}}{Q_{(m)}} \right)^{-1}. \hfill (57)$$

The ratio of the output power to PT kinetic energy stored characterizes the PT vibrational efficiency

$$\frac{P_R}{\omega W_{kin}} = \frac{1}{Q_{sys}} - \frac{1}{Q^\ast_{(m)}} = \eta - \frac{1}{Q_{(m)}} = 0.5 B_3 k_{33}^2 \cdot \frac{t}{1 + r^2}, \quad \hfill (58)$$

so that its maximum value $\left(\frac{P_R}{\omega W_{kin}}\right) = 0.1$ under maximum efficiency loading.

Note the difference between two groups of the quality factors: $Q_{sys}$ and $Q^\ast_{(m)}$ with included the dielectric losses of the input PT section, while $\tilde{Q}$ and $Q_{(m)}$ are not under dielectric loss influence (at least of the input PT section). Respectively, there are two basic practical methods of quality factor experimental determination. According to (12,13), a method based on amplitude measurements [10] of the PT resonance admittance $|Y_{in}|$, or PT conductance $\text{Re} Y_{in(r)}$, takes into account the dielectric losses; a method based on susceptance $\text{Im} Y_{in(f)}$ extreme frequencies measurement [32] are not under influence of the dielectric losses. For an elementary unloaded piezoceramic resonator the dielectric loss factor typically $\delta_{33} \ll k_{23}^2 Q_{(m)}$, and the quality factors difference is extremely low. However, for a loaded and/or strong-field excited PT, the dielectric loss factor greatly increases while the system’s vibrational quality factor reaches as low as $\sim 10$ units – in such a situation the mentioned quality factors difference is significant.

An energetic derivation of the PT transfer function is presented further to demonstrate physical nature of the quality factor components included. As the input and output PT sections are under different electrical
conditions, their intrinsic dissipative material parameters are different. Considering a free-vibrating
asymmetric PT plate (Fig. 1,2), from the momentum conservation principle \( \int x \bar{v} \, dx = 0 \) for a whole body,
where \( \bar{v} = iomega \vec{u} \) is the velocity, at least for a Langevin-type PT with equal masses we have equality of kinetic
energies of the input and output PT sections \( W_{kin(1)} = W_{kin(2)} = 0.5W_{kin} \). Then, the material quality factor \( Q_{(m)} \)
of the PT as a whole can be expressed:

\[
Q_{(m)} = \omega \sum W_{kin} \sum \frac{W_{kin(1)} + W_{kin(2)}}{P_v + P_{v(2)}} , \quad \text{then} \quad \frac{1}{Q_{(m)}} = \frac{1}{Q_{m(1)}} + \frac{1}{Q_{m(2)}} ,
\]

(59)

where \( Q_{m(i)} = \omega W_{kin(i)} / P_{v(i)} \) is the intrinsic material quality factors of the input \( (i = 1) \) and output \( (i = 2) \) PT
sections with transverse and longitudinal vibrational modes [28,32], respectively. Finally, the vibrational PT
quality factor \( \tilde{Q} \) is determined as

\[
\tilde{Q} = \omega \sum W_{kin} \sum \frac{W_{kin(1)} + W_{kin(2)}}{P_v + P_{v(2)} + P_R} , \quad \text{then} \quad \frac{1}{\tilde{Q}} = \frac{1}{Q_{m(1)}} + \frac{1}{Q_{m(2)}} + \frac{P_R}{\omega W_{kin}} .
\]

(60)

The ratio of the output power to PT kinetic energy characterizes PT material piezoactivity, or a degree of
output PT section polarization. It can be estimated by considering the equivalent representation of the PT
output section under an electrical load, shown in Fig. 13. As the PT piezoelectric output can be represented by
the “electro-mechanical” electromotive force (EMF) \( \tilde{E} \), then the PT output voltage and power are:

\[
U = J R = \tilde{E} \frac{R}{1/iomega C_{(out)}} + R , \quad P_R = 0.5U \omega \tilde{E} \cdot \frac{Romega C_{(out)}}{1 + (Romega C_{(out)})^2} \left\{ \max P_{el} \right\} \frac{t}{1 + t^2} ,
\]

(61)

\( \text{Fig. 13. Equivalent representation of the PT output section “EMF} \)

\( \tilde{E} \) - output output capacitor \( C_{(out)} \) “under electrical load \( R \).
where \( t = R_\omega C_{(out)} \), \( \{ \max P_{el} \} = 0.5 \vec{E} \cdot \omega C_{(out)} \) is the averaged for a period maximum electric power created by the generator \( \vec{E} \) (corresponds to a “short-circuit” regime), then \( \{ \max W_{el} \} = \{ \max P_{el} \}/\omega \) is the maximum electric energy provided by the generator. According to a CEMC definition applied to the PT output section, \( k^2_{31} = \{ \max W_{el} \}/W_{kin(2)} \), then the ratio of the PT output power to kinetic energy stored:

\[
\frac{P_g}{\omega W_{kin}} = 0.5 \frac{\{ \max W_{el} \}}{W_{kin(2)}} \frac{t}{1 + t^2} = 0.5 k^2_{33} \frac{t}{1 + t^2},
\]

(62)

The last expression (62) is not taking into account the plate “node” displacement and, hence, related to it \( B_3 \) (output PT section) coefficient. Also, for one dimensional vibration modes, longitudinal or transverse, the dynamic coupling factor is related to the appropriate material coupling factor by the proportionality coefficient \( 8/\pi^2 \cong 0.81 \), that takes into account the sinusoidal variables distribution [34].

**B. Non-linear PT Behavior: Quality Factor and Characteristics of a PT under High Level of Excitation**

In general, PTs exhibit nonlinearity as well as temperature dependence – both these effects restrict the PT performance. Consequently, the model parameters need to be extracted under the intended operating conditions. Well known effects at high excitations (strong electrical and/or mechanical fields), such as the drastic mechanical material quality factor \( Q_m \) decreasing up to an order, dielectric losses increasing, and the resonance and antiresonance frequency shifting up to several percent, are taken into consideration. Some of the characteristics primary depend on the electric field strength (dielectric loss), others mostly on mechanical stress (quality factor, resonance frequencies). A semi-empirical approach was involved, instead of the common case when non-linear constitutive piezoelectric equations should be used. Non-linear experimental data preliminary determined in the process of piezoceramic material characterization on standard samples [31] were used to describe a real PT by iterative procedure on the basis of quality factor concept developed here.

The input electric voltage governing a PT is a primary parameter that is specified and easy controlled. For this reason, the following statement was used to link strong-field data received for different vibrational conditions. Applied to a standard bar-shape piezoceramic sample, the electric voltage \( \vec{V} \) creates the electric
field strength $\bar{E}$ (including non-resonance frequencies) and effective mechanical stress $\bar{T}$ at the resonance. The material quality factor $Q_{(m)}(\bar{T})$ and dielectric loss factor $\delta_{33}(\bar{E})$ are supposed to be predetermined in the process of piezomaterial characterization. When a bar-shape PT is excited by the same voltage $\bar{V}$ (strength $\bar{E}$) and is electrically loaded, its system’s quality factor $\bar{Q}$ and proportional to it mechanical stress $T \equiv \bar{T} \cdot \bar{Q}(T)/Q_{(m)}(\bar{T})$ greatly decrease due to the loading, in turn influencing the material quality factor value. Then, the PT material parameters under such new conditions are $Q_{(m)}(T)$, while $\delta_{33}(\bar{E})$ remains the same corresponding to the $\bar{E}$-field.

Typically, the strong-field material quality factor $Q_{ms}$ dependence is determined through the resonance resistance $R_r$ variation on excitation level on standard resonators such as a bar, disc, or ring [31]. Due to specific mechanical stress distributions for such vibrational modes, they have slightly different strong field characteristics dependences [21], so correction coefficients must be used. According to the literature [19-21], the strong field dependence in common case can be expressed as an exponential function:

$$R_r(\bar{E})/R_{r0} \equiv \frac{Q_{m0}}{Q_{ms}} \equiv \left\{ 1 + \left( \frac{\bar{E}}{E_{Q1}} \right)^2 + \left( \frac{\bar{E}}{E_{Q2}} \right)^2 + \ldots \right\},$$

(63)

where $Q_{ms} \equiv Q_m(\bar{E})$, $Q_{m0} \equiv Q_{(m)}(\bar{E} \to 0) \equiv Q_m$, $E_{Qp}$ is the electric field strength threshold $(p = 1,2,3\ldots)$ for a quality factor, determined experimentally at material characterization on some standard piezoresonator with elementary vibrational type. In common, the mechanical stress $T = \beta_{(j)} \bar{E} Q_m(T)$ (locally, as well as integrally on the resonator’s major surface), where $\beta_{(j)} \sim a_{(j)} k_{(j)}^2$ is a piezo-elastic coefficient proportional to respective CEMC squared. For a particular resonator, the expression

$$\frac{Q_m(T)}{Q_{m0}} = \sum_{p=0}^{\infty} \left( \frac{\bar{E}}{E_{Qp}} \right)^p = \sum_{p=0}^{\infty} \left( \frac{1}{\beta_{(j)} E_{Qp} Q_m(T)} \right)^p$$

is supposed to be independent on vibrational type, then it follows that $\beta_{(j)} E_{Qp} = const$ is independent on the type of vibration. From this we can particularly
conclude that for bar and disk resonators: \[ \frac{E_{0_{p(bar)}}}{E_{0_{p(disk)}}} = \frac{\beta_{(disk)}}{\beta_{(bar)}} = \frac{2}{1-\sigma} = \frac{k_p^2}{k_{31}^2} = 3 \], where \( \sigma \) is the Poisson coefficient. Experimentally determined \( E_{Q1(disk)} = 1.6 \text{ V/mm} \) for a typical disk piezoresonator \( \varnothing 20 \times 1 \text{ mm} \) made from hard PT piezomaterial, and for a bar resonator \( E_{Q1(bar)} = 5 \text{ V/mm} \) was found as a first-order (linear) slope of the strong field resonance admittance dependence.

We can use the expression (7) for the load-dependent PT quality factor, applying it to high level drive \( \frac{1}{Q} = \frac{1}{Q_{ms}} + 0.5 B_2 k_{33}^2 \frac{t}{1+t^2} \),

where \( Q_{ms} \) is the material quality factor in strong regime of excitation. To satisfy the condition of equal mechanical stress \( T \) for an unloaded and electrically loaded bar PT:

\[ \bar{V} \cdot Q_{ms} = V \cdot \tilde{Q} \]

(it can be called as a PT non-linear “voltage-reduced effect under load”), then from (63) follows

\[ \frac{Q_{m0}}{Q_{ms}} = 1 + \sum_{p=1}^{\infty} \left( \frac{V}{hE_{Q,p}} \right)^p \left( \frac{\tilde{Q}}{Q_{ms}} \right)^p = 1 + \sum_{p=1}^{\infty} \left( \frac{V}{hE_{Q,p}} \right)^p \left[ 1 + B_2 k_{33}^2 \frac{t}{2(1+t^2)} \right]^{-p} \],

which can be solved numerically by the iterative procedure to receive material quality factor \( Q_{ms}(V) \) value corresponding to a given level of PT excitation, and then system’s vibrational quality factor \( \tilde{Q}(V) \) under load.

According to experimental data, there is a relatively small contribution for practical PT excitation levels from the second-order term in (63), as typically \( E_{Q2} > 5 E_{Q1} \), so that its influence is not a predominant factor especially for hard PT piezomaterials [19]. To illustrate the proposed approach, particularly for \( p = 1 \), equation (66) can be solved with a simple analytical procedure as follows:

\[ \left( \frac{Q_{m0}}{Q_{ms}} \right)^2 - (c_1 - c_2) \left( \frac{Q_{m0}}{Q_{ms}} \right) - c_2 = 0 \], \quad \text{then} \quad (67)

\[ 2 \left( \frac{Q_{m0}}{Q_{ms}} \right) = c_1 - c_2 + \sqrt{(c_1 - c_2)^2 + 4 \cdot c_2} \], \quad (68)
Fig. 14. Calculated (64-66) "strong field" material and vibrational PT quality factors vs. relative load parameter \( t = \omega_{(r)} C_2 R \) of the Langevin-type PT for different levels of excitation. Material parameters used:

\[
k_{33} = 0.65, \quad B = s_{33}^E / s_{11}^E = 1.30, \quad Q_{m0} = 500, \quad E_{Q1} = 5 \text{ V/mm}.
\]

Fig. 15. Non-linear effects in Langevin-type PT output voltage and input power characteristics: curves for \( t = \infty \) corresponds to o.c., \( t = 1 \) – maximum efficiency regimes; points a – initial point (ex., ignition), b – operational point; coefficients \( K_0 = \max |U_{r|_{t=\infty}}| / L \) (69), \( K_1 = 0.5 \cdot 10^4 \omega_{(r)} \varepsilon_{33}^T \text{ wLH} \),

\[
K_2 = 0.2 k_{33}^2 Q_{m0} \approx 50.
\]

Material parameters used: \( k_{33} = 0.65, \quad k_{31} = 0.32, \quad B = s_{33}^E / s_{11}^E = 1.30, \quad Q_{m0} = 500, \quad E_{Q1} = 5 \text{ V/mm}, \quad \delta_{33(0),0} = 0.01, \quad E_{\delta 1} = 25 \text{ V/mm} \).
where \( c_1 = 1 + \frac{V}{h E_{Q_1}} \), \( c_2 = 0.5 B_2 \ k_{33}^2 Q_{m0} \cdot \frac{t}{1 + t} \) are coefficients. In Fig. 14 the material and PT vibrational quality factor dependences are shown for relatively low and high levels of PT excitation. As a consequence of such a non-linear effect, the output voltage of an o.c. PT \( U \sim (V/h) \cdot Q_{m_s} \) reaches saturation (Fig. 3):

\[
\max \frac{|U_r|_{t \to \infty}}{L} = \sqrt{\frac{k_{33}^E}{s_{31}^E + s_{33}^E}} \ k_{31}^E |Q_{m_0} \cdot E_{Q_1}|, \tag{69}
\]

which can be considered as a non-linear material limit of the voltage gain. Physically it occurs when the value of \( Q_{m_s} \) is determined predominantly by a voltage-dependable term in (64). This is a non-linear, not a thermal, effect. A level of 20 % of the \( \max |U_r| \) saturation takes place at \( V/h \equiv 4 E_{Q_1} \) (Fig. 15). Note, that the second-order term in (63) provides a maximum in the output PT voltage field-dependence.

For maximum PT efficiency regime with \( t = 1 \), the saturation level \( \sim 0.6 \max |U_r|_{t \to \infty} \) occurs at extremely high electric field strength, a 0.5-level of the saturation corresponds to \( V/h = 0.2 k_{33}^2 Q_{m0} \cdot E_{Q_1} \), as it follows from (18) and (68), and is shown in Fig. 15. The initial I/O slopes for both o.c. and max efficiency regimes for low-level excitation are determined by (17-20). As to practical operational conditions, a very high voltage is required to initially ignite a lamp, while during sustained operation the voltage requirements are significantly reduced. So, the transition from initial ignition point \( (a) \) to operational point \( (b) \) can be easily realized using a power supply source with finite internal resistance, taking into account that PT input resonance impedance greatly changes under load up to \( 0.25 k_{33}^2 Q_{m0} \approx 50 \) times.

The dielectric loss and elastic compliance non-linearities can be taking into account as follows:

\[
\frac{\delta_{33,t} (V)}{\delta_{33,0}} = 1 + \sum_{p=1}^{\infty} \left( \frac{V}{h E_{p}} \right)^p, \quad \frac{\delta_{ij,t} (V)}{\delta_{ij,0}} = 1 + \sum_{p=1}^{\infty} \left( \frac{V}{h E_{s_p}} \right)^p = 1 + \sum_{p=1}^{\infty} \left( \frac{V}{h E_{s_p}} \right)^p \cdot \left( \frac{Q}{Q_{ms}} \right)^p, \tag{70}
\]

where \( E_{p} \) and \( E_{s_p} \) are the respective electric field thresholds determined experimentally on some standard piezoresonator at material characterization. Effects of such non-linearities are demonstrated in Figs. 4,7,8,15.

Similar approach and iterative procedure can be applied for describing temperature PT characteristics.
5. CONCLUSIONS

A theoretical understanding of the PT performance has been developed through the quality factor concept. A simple 1-D classical model of the Langevin-type PT was considered analytically in details and the effective system’s PT quality factor, as a basic resonance and near resonance PT parameter, was analyzed. Then, the generalized results were applied to the Rosen-type PT as most practical. The electrical loading, partly short-circuiting the output PT section, changes PT effective system’s vibrational (dynamic) complex compliance, whose real component influences the operational PT resonance frequency, and its imaginary part, related to the system’s vibrational quality factor, takes into account losses in each PT input-output-load functional components. As the PT quality factor and operational frequency dependences on the PT load are separately established, they can be used for characterizing the PT performance in a wide frequency range, including open-circuit, short-circuit, and maximum efficiency conditions, taking into account non-linearities of dissipative, elastic and piezoelectric material parameters under high vibrational level of excitation.

Under constant on frequency input voltage applied to PT, maximum output power corresponds to the condition of equal PT thermal and output losses; PT maximum efficiency regime corresponds to the minimum PT thermal losses, or minimal PT vibrational quality factor. The quality-factors ratio, between the state of non-loaded and fully-loaded PT, is one of the best measure of PT’s dynamic performances. Practically, the higher the ratio between non-loaded and fully-loaded PT quality factors is, the better PT efficiency. A simple and effective method of the loaded PT quality factor determination is proposed.

As was found, a piezoceramic with high piezoelectric anisotropy, and hence with higher Poisson coefficient of non-polarized piezoceramics, is required to provide maximum PT efficiency and higher corresponding voltage gain. The basic PT characteristics are greatly sensitive to strong field non-linearities of the material dissipative parameters: material quality factor degradation restricts maximal PT output voltage, and the dielectric loss rising leads to a “flat” PT efficiency characteristic with a shift of optimal practical load value up to 8. The basic PT resonance frequency interval (o.c. – s.c.), depending on material elastic anisotropy, is almost twice lower than that for an elementary rod resonator with a $k_{33}$-type vibrational mode, and does not depend on the material CEMC $k_{31}$ of the input PT section.
The generic closed-form formulas developed in this study reveal some universal relationships between key parameters of a PT, and these physical trade-offs can be used two ways: to optimize the design of a PT for a given application and/or specifying the desired PT to fill given tasks. The presented approach, as a new contribution into multi-electrode piezoelectrics characterization, is high promising and can be applied to other voltage- and current PT types for its temperature performance and frequency control optimization.

REFERENCES


http://www.ucop.edu/research/micro/96_97/96_032.pdf


Glossary

\( Q_m, Q_{m(s)} \) – standardized piezomaterial quality factor for low and strong field of excitation
\( Q^{E,D}_{ij}, Q_{sys} \) – material quality factors of the complex elastic compliances \( S^{E,D}_{ij} \)
\( \hat{Q}, Q_{sys} \) – system’s vibrational electro-mechanical PT quality factor and total PT quality factor
\( \gamma_{33} (\gamma_{31}), \gamma_{33} (\hat{\gamma}_{31}) \) – dielectric and piezoelectric loss factors
\( k_{ij} (k_{ij}), \hat{k}_{ij} \) – coefficient of electro-mechanical coupling (complex and real values)
\( \hat{r}, \hat{r}_{r,a} \) – relative resonance frequency interval of a standard piezoelement [31]
\( \{G_0, G_{\infty}, G_{eff}, G_{r,r}\} \) – relative PT resonance frequency interval
\( \chi_r, \chi_y \) – relative and generalized relative resonance frequency displacement
\( C_{1(in)}, C_{2(out)} \) – PT capacitance (real values)
\( s_1, s_2, s \) – dynamic PT compliance of input section, output section and total PT under electric load
\( \alpha, B, B_1, B_2 \) – parameters of material and PT elastic anisotropy
\( A \) – parameter of piezoelectric anisotropy
\( H, h, n \) – PT plate thickness, thickness of a layer, and number of layers
\( M \) – PT mass load (for Langevin-type PT model)
\( x \) – PT relative local coordinate [-1:1]
\( V, I, Y (Z) \) – input power supply voltage, current and PT admittance (impedance)
\( U, J, R, t \) – PT output voltage, load current, load resistance and relative load resistance
\( D_{(3)}, E_{(3)}, T_{(1,3)}, S_{(1,3)} \) – electric field induction and strength, mechanical stress and strain
\( u, u_m, u_0, u_m \) – local, PT top, PT center and minimal displacements
\( w, L, l \) – PT width, half-plate length and PT frequency-specifying dimension
\( x_m \) – position of the PT point with “zero” (minimal) displacement
\( W_{hin} \) – kinetic PT mechanical energy stored
\( J_{in}, P_R, P_h, \eta \) – input, output and PT thermal (heat) loss power, and PT efficiency
\( E_{Q1}, E_{\alpha1} \) – material first-order electric field parameters of quality factor and dielectric loss factor non-linearities

\[
\begin{align*}
\frac{s_1}{s} & \rightarrow S^{E}_{11}, s_2 & \rightarrow S^{E}_{33} \left[ 1 - k_{33}^2 \frac{t^2}{1 + t^2} \right], \\
& \equiv 0.5 \left( s_1 + s_2 \right) = 0.5 \left[ \frac{S^{E}_{11}}{s} + \frac{S^{E}_{33}}{s} \left( 1 - k_{33}^2 \frac{t^2}{1 + t^2} \right) \right], \\
B_1 & = \frac{S^{E}_{11}}{S^{E}_{33}}, B_2 = \frac{S^{E}_{33}}{S^{E}_{11}}, B & \equiv \frac{B_2}{B_1} = \frac{S^{E}_{33}}{S^{E}_{11}}, \alpha \equiv \frac{s_2}{s_1} & \rightarrow B \left[ 1 - k_{33}^2 \frac{t^2}{1 + t^2} \right],
\end{align*}
\]

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